Comparison of Markov and Least-Squares Missile Position and Velocity Estimates

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Introduction

AN increase of precision in missile position and velocity estimates can be obtained by using smoothing and rate filters that consider the serial (time) correlation of the errors in the measurements of a radio tracking system.

We will consider the estimation of the position and velocity vector of a vehicle that is observed by a radio tracker that provides three coordinates as functions of the time. The following considerations generally apply to the case of a radio tracking system measuring range, azimuth, and elevation or range and two direction cosines or range and two range differences. However, our numerical example corresponds to the AZUSA Mark II continuous wave (CW) radar, for which there is available trajectory and error structure data. First we will proceed to compare minimum variance (Markov) smoothing and first-derivative filters with least-squares filters.

In order to make a simple comparison, we will apply the continuous theory of Zadeh and Ragazzini[1] rather than the digital filters that can be constructed by algebraic means[2–6] in the sampled-data case. Under very general conditions, Swerling[7] has shown that this optimum linear estimate based on continuous observation during a finite time T is the limit of optimum linear estimates based on sampled data as the sample spacing tends to zero. Since the sample spacing of AZUSA Mark II is $\frac{1}{20}$ sec, we will assume that these continuous estimates are sufficiently close to the discrete estimates for the sampled-data case. Thus we will be concerned with the calculation of the ratio

$$\mathfrak{F} = [\langle \epsilon_{ls}^2 \rangle / \langle \epsilon_{op}^2 \rangle]^{1/2} \qquad (1)$$

where $\langle \epsilon_{ls}^2 \rangle$ is the output mean-squared error of the least-squares continuous filter, and where $\langle \epsilon_{op}^2 \rangle$ is the output mean-squared error of the optimal continuous (minimum variance) filter. In the case of smoothing filters the ratio $\mathfrak{F}$ will be designated as $\mathfrak{F}$; in the case of first-derivative filters as $\mathfrak{F}_d$.

Statement of the Smoothing and First-Derivative Filtering Problem

We assume[1] that the unfiltered position data $d(t)$ are a known real function of time t, $0 \leq t \leq T$:

$$d(t) = y(t) + n(t) \qquad y(t) = g(t) + m(t) \qquad (2)$$

where $n(t)$ is the measurement error, and where $g(t)$ is a deterministic polynomial function of time, and $m(t)$ is the random component of the signal $y(t)$. The degree of $g(t)$ is r, and $m(t)$ and $n(t)$ are stationary ergodic and uncorrelated random functions of time with mean values equal to zero. With the assumption that data concerning $g(t)$, $m(t)$, and $n(t)$ are only known during the finite interval $(0, T)$, the best linear estimate $x(t)$ of $y(t)$ is defined by

$$x(t) = \int_0^T \{g(t-\tau) + m(t-\tau) + n(t-\tau)\} k(\tau) d\tau \qquad (3)$$

where $k(\tau)$ is the optimal weighting function or impulse response characterizing the filter.

If $n(t)$ and $m(t)$ have a Gaussian distribution, $x(t)$ is a maximum likelihood estimate. When the distributions are non-Gaussian, $x(t)$ is a linear minimum variance (Markov) estimate.

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Optimum Filter Weighting Functions

In order to simplify these calculations, we will assume that the random component $m(t)$ of $y(t)$ is identically zero, that $g(t)$ is a first-degree polynomial so that $r = 1$, and that the correlation function $\mathfrak{R}_n(\tau)$ and the spectral density of the errors $n(t)$ are given by

$$\mathfrak{R}_n(\tau) = \sigma^2 e^{-a|\tau|} = (K_1 a/2) e^{-a|\tau|} \qquad \sigma^2 = K_1 a/2 \qquad (4)$$

$$S_n(\omega) = K_1 a^2/(\omega^2 + a^2) \qquad (5)$$

Both $\mathfrak{R}_d(\tau)$ and its Fourier transform $S_n(\omega)$ are a sufficient approximation for the representation of radio tracking error structures.

The filter error variance is given by

$$\langle \epsilon^2 \rangle = \int_0^T \int_0^T \mathfrak{R}_n(\tau - \theta) k(\tau) k(\theta) d\tau d\theta \qquad (6)$$

Since $x(t)$ must be an unbiased estimate, two constraining conditions on the weight function $k(\tau)$ are determined in terms of its moments:

$$\mu_\nu = \int_0^T \tau^\nu k(\tau) d\tau \qquad \nu = 0, 1 \qquad (7)$$

The variational problem of minimizing the functional in Eq. (6) [where $\mathfrak{R}_n(\tau)$ is the autocorrelation function of $n(t)$] under the constraints of Eq. (7) has been solved by Zadeh and Ragazzini.[1] The necessary and sufficient condition for a minimum of $\langle \epsilon^2 \rangle$ is that the function $k(t)$ satisfy the integral (Euler) equation

$$\int_0^T \mathfrak{R}_n(t - \tau) k(\tau) d\tau = \gamma_0 + \gamma_1 t \qquad (8)$$

where γ_0, γ_1 are the Lagrange multipliers in the associated functional

$$I\{k\} = \langle \epsilon^2 \rangle - 2\gamma_0 \mu_0 - 2\gamma_1 \mu_1 \qquad (9)$$

where μ_0, μ_1 are the integral constraints defined by Eqs. (7).

The integral equation (8) can be reduced to a differential equation whose general solution under the preceding assumptions leads to the optimum function[1]

$$k(t) = A_0 + A_1 t + C_1 \delta(t) + D_1 \delta(t - T) \qquad 0 \leq t \leq T \qquad (10)$$

where $\delta(t)$ is Dirac's ideal function. By substituting Eq. (10) in Eq. (8) and by quadratures and Eqs. (7) and (8), four linear equations follow. These linear equations are relative to A_0, A_1, C_1, D_1 with coefficient functions of a, T, μ_0, and μ_1. Formulas for A_0, A_1, C_1, D_1 in terms of a and T are given [see Eqs. (79–82) of Ref. 1] in Ref. 1 for both the smoothing ($\mu_0 = 1$, $\mu_1 = 0$) and first-derivative filtering cases ($\mu_0 = 0$, $\mu_1 = -1$). The minimum mean-square output error in the smoothing case is

$$\langle \epsilon_{ops} \rangle = \frac{8\sigma^2}{a} \left[\frac{[T^2 + (3T/a) + (3/a^2)]}{[T^2 + (6T/a) + (12/a^2)][T + (2/a)]} \right] \qquad (11)$$

and in the first-derivative filtering case is

$$\langle \epsilon_{opd}^2 \rangle = \frac{24\sigma^2}{a[T^2 + (6T/a) + (12/a^2)]T} \qquad (12)$$

Characterization of Least-Squares Filters and Their Output Errors for Correlated Input Errors

In the case of uncorrelated $n(t)$, the coefficients of the optimum weight function for the smoothing filter are obtained by taking the limits of Eqs. (79–82) of Ref. 1 for $a \to \infty$. Thus

$$A_0 = 4/T \qquad A_1 = -(6/T^2) \qquad C_1 = 0 \qquad D_1 = 0 \qquad (13)$$

and, for the first-derivative filter,

$$A_0 = \frac{6}{T^2} \qquad A_1 = -\frac{12}{T^3} \qquad C_1 = 0 \qquad D_1 = 0 \quad (14)$$

The filters defined by Eqs. (13) and (14) are thus least-squares filters because they are optimal for uncorrelated errors. When the errors $n(t)$ are correlated and defined by Eqs. (4) and (5), the output mean-square errors $\langle\epsilon_l^2\rangle$ of the least-squares filters characterized by Eqs. (6, 13, and 4) will not be of minimum variance. By Eqs. (13) and (14) we have $k_l(t) = A_0 + A_1 t$, and by Eqs. (4) and (6) it follows that

$$\langle\epsilon_l^2\rangle = \sigma^2 \left\{ A_0^2 \left[\frac{2T}{a} - \frac{2}{a^2} + \frac{2e^{-aT}}{a^2}\right] + 2A_0A_1\left[\frac{T^2}{a} - \frac{T}{a^2} + e^{-aT}\frac{T}{a^2}\right] + A_1^2\left[\frac{2}{a^4} + \frac{2}{3}\frac{T^3}{a} - \frac{T^2}{a^2} - 2e^{-aT}\left(\frac{T}{a^3} + \frac{1}{a^4}\right)\right]\right\} \quad (15)$$

For the smoothing filter we find, by Eqs. (13) and (15),

$$\langle\epsilon_{ls}^2\rangle = \sigma^2 \left\{ \frac{8}{(aT)} - \frac{20}{(aT)^2} + \frac{72}{(aT)^4} - e^{-aT}\left[\frac{16}{(aT)^2} + \frac{72}{(aT)^3} + \frac{72}{(aT)^4}\right]\right\} \quad (16)$$

For the first-derivative filter we have, by Eqs. (14) and (15),

$$\langle\epsilon_{ld}^2\rangle = \sigma^2 a^2 \left\{ \frac{24}{(aT)^3} - \frac{72}{(aT)^4} + \frac{288}{(aT)^6} - \frac{72}{(aT)^4}\left(1 + \frac{4}{(aT)} + \frac{4}{(aT)^2}\right)e^{-aT}\right\} \quad (17)$$

Using Eqs. (11, 16, 12, and 17), we can form the ratios

$$\mathfrak{F}_s = \left[\frac{\langle\epsilon_{ls}^2\rangle}{\langle\epsilon_{ops}^2\rangle}\right]^{1/2} = \mathfrak{F}_s(aT)$$
$$\mathfrak{F}_d = \left[\frac{\langle\epsilon_{ld}^2\rangle}{\langle\epsilon_{opd}^2\rangle}\right]^{1/2} = \mathfrak{F}_d(aT) \quad (18)$$

which give the gains in precision of Markov filters over least-squares filters.

Variance of Position and Velocity Estimates for the AZUSA Mark II Tracking System in the Case of Markov or Least-Squares Filtering

The measurement digital data output of AZUSA Mark II corresponds to range and two direction cosines. The orthogonal coordinates x, y, z are related to the measurements coordinates r, l, m by

$$x = rl \qquad y = rm \qquad z = rn \quad (19)$$

where r is the range, and l, m, and $n = (1 - l^2 - m^2)^{1/2}$ are direction cosines.

By Eq. (19) and assuming that the r, l, and m measurement error processes are not cross-correlated, the dispersions

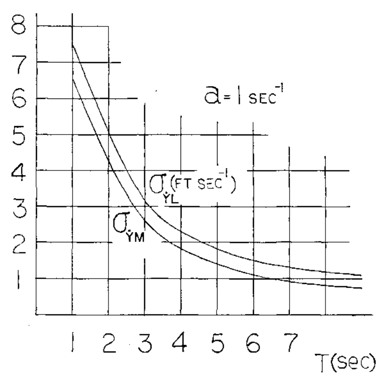

Fig. 1 Markov and least-squares standard deviation $\sigma_{\dot{y}}$.

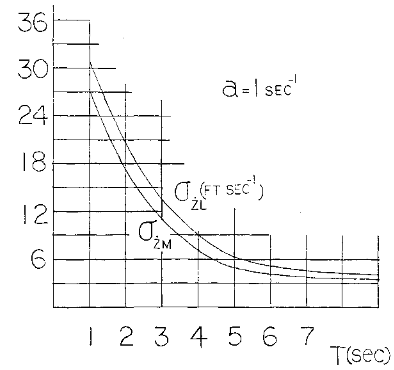

Fig 2. Markov and least-squaes standard deviaton $\sigma_{\dot{z}}$.

σ_x, σ_y, σ_z, $\sigma_{\dot{x}}$, $\sigma_{\dot{y}}$, $\sigma_{\dot{z}}$ of the position and velocity estimates in coordinates x, y, z are given by well-known formulas that follow linear transformation theory.

To perform the comparison, we take the following numerical values[8]: $r = 2{,}482{,}865$ ft, $l = -0.7370799787$, $m = -0.6587986102$, $n = 0.1506568823$, $\dot{r} = 17{,}109.0$ ft-sec^{-1}, $\dot{l} = -0.0542065690 \times 10^{-3}$ sec^{-1}, $\dot{m} = -0.2157745458 \times 10^{-3}$ sec^{-1}, $\dot{n} = -0.1208750273 \times 10^{-2}$ sec^{-1}, $\sigma_r^2 = 49$ ft^2, $\sigma_l^2 = (1.5)^2 \times 10^{-12}$, $\sigma_m^2 = (1.5)^2 \times 10^{-12}$, and $a = 1$ sec^{-1}. The updated (in "real time" at $t = T$) Markov and least-squares estimates are denoted with the subscripts M and L, respectively.†

Discussion of Numerical Results

The gains $\mathfrak{F}$ in precision of the Markov over least-squares estimation techniques are indicated by Figs. 1 and 2 as functions of a and the filtering time T. (Since the values $\mathfrak{F}_{\dot{x}} = \sigma_{\dot{x}L}/\sigma_{\dot{x}M}$ are rather close to those of $\mathfrak{F}_{\dot{y}}$, they are not represented in graphical form.)

The numerical results indicate that greater gains can be expected in the case of derivative filtering than with position smoothing. The computed precision increments in the estimation of position parameters ($\mathfrak{F}_x$, $\mathfrak{F}_y$, $\mathfrak{F}_z$) are about 11%. However, in the derivative-filtering case, the precision gains vary by a factor $\mathfrak{F}_d = 1.129$ to $\mathfrak{F}_d = 1.25$, depending on the filtering time T. With a linear approximation to a typical rocket trajectory, the velocity estimates would be strongly biased if T were not limited to $T \leq 5$ sec.‡

More general and complete conclusions can be attained by an analysis that considers given nominal trajectories and more general regression and error structures.

Conclusions

The results of this analysis indicate the advantage of optimal (Markov) filtering over least-squares filtering, although this is based on the hypothesis of a linear approximation to the trajectory during the observation time T.

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† By considering appropriate changes in the moment equations, it can be shown that the output variance of the Markov smoothing filter (for a linear signal) at the midpoint $t = T/2$ of $[0, T]$ is one-fourth of the variance at the endpoint $t = T$. For the Markov derivative filter and a linear signal, the output variance is independent of the estimation point in $[0, T]$.

‡ A value of T of 5 sec is of current use at the Air Force Missile Test Center.[9] The z axis is in the vertical direction.

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A Compatibility Equation for Nonequilibrium Ionization

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AN expression of compatibility, independent of the electron collision frequency, is developed from the electron heating equation (discussed in Ref. 1) and the Saha equation. By using this compatibility relation, one may compute the profiles of electron temperature, plasma electrical conductivity, and electric field, between the anode and cathode of moderate pressure plasma devices (such as a diode or an MHD generator), once the neutral particle temperature profile and the current density have been measured. The current passing through any plane parallel to the electrodes is assumed constant. The influence of nonuniformities due to insulator walls is neglected. Radiation losses are neglected. As a first approximation, electron diffusion is also neglected.

Starting with the electron heating equation in which very steep spatial gradients are neglected,

$$j^2/\sigma = n_e\nu_e(2m_e/m_n)\delta(\tfrac{3}{2}kT_e - \tfrac{3}{2}kT_n) \tag{1}$$

where j is the current density, σ the plasma scalar electrical conductivity, n_e the electron density, ν_e the electron collision frequency, and m_e the electric mass. m_n is the average heavy particle mass ($2m_e/m_n$ represents the average fraction of energy loss per electron per elastic heavy particle collision), δ the loss factor that accounts for inelastic collisions and radiation losses (δ is near unity for monatomic particles but is very large for polyatomic particles, as discussed in Ref. 2), k the Boltzmann constant, T_e the average electron temperature, and T_n the average neutral particle temperature.

The scalar conductivity equation is

$$\sigma = (n_e e^2/m_e\nu_e) \tag{2}$$

where e is the electron charge.

The electron collision frequency is eliminated by combining Eqs. (1) and (2). The resulting expression is then substituted into the Saha equation:

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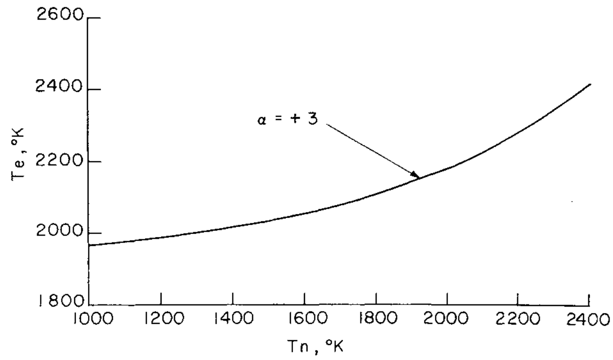

Fig. 1. Electron temperature vs neutral particle temperature for $\alpha = 3$.

$$\frac{n_e^2}{n_s} = \left(\frac{2\pi m_e kT_e}{h^2}\right)^{3/2} e^{-E_0/kT_e} \tag{3}$$

where n_s is the number density of ionizable particles (in the case of a seeded plasma, $n_s = X_s n_n$, where X_s is the mole fraction of seed in n_n neutral particles), h is the Planck constant, and E_0 is the ionization potential. Note that Eq. (3) is valid when $n_e \ll n_s$ and when the ratio of the statistical weights is equal to unity.

This resulting expression (the compatibility equation) is found to be

$$j^2 = \left(\frac{3_e{}^2\delta}{m_n}\right)(X_sP_n)\left(\frac{T_e}{T_n} - 1\right)\left(\frac{2\pi m_e kT_e}{h^2}\right)^{3/2} e^{-E_0/kT_e} \tag{4}$$

For convenience of computation, Eq. (4) is rearranged:

$$\frac{5040\ E_0}{T_e} - \log\left(\frac{T_e}{T_n} - 1\right) - 1.5\log T_e = 8.05107 - \alpha \tag{5}$$

where

α = $\log(j^2M_n/\delta X_sP_n)$
j = current density, amp/cm²
M_n = average neutral particle atomic weight
P_n = static pressure, atm
T_n = neutral particle static temperature, °K
T_e = average electron temperature, °K
E_0 = ionization potential, ev

Consequently, when α and $T_n(x)$ are measured, $T_e(x)$ can be computed from Eq. (4). (For convenience, a plot of T_n

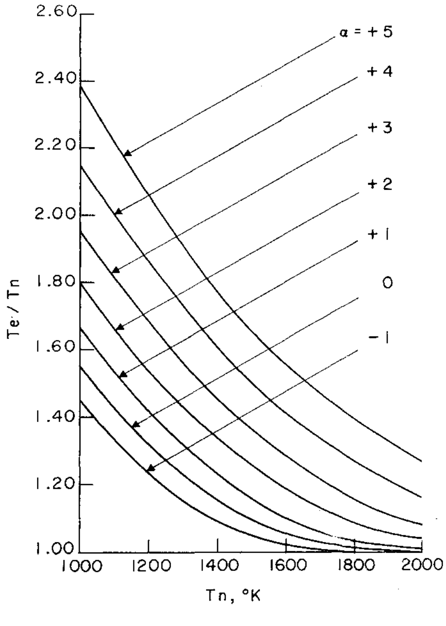

Fig. 2 Temperature ratio vs neutral particle temperature for various α.